MODELING OF THE PROPAGATION OF SEISMIC WAVES IN NON-CLASSICAL MEDIA
Reduced Cosserat Continuum

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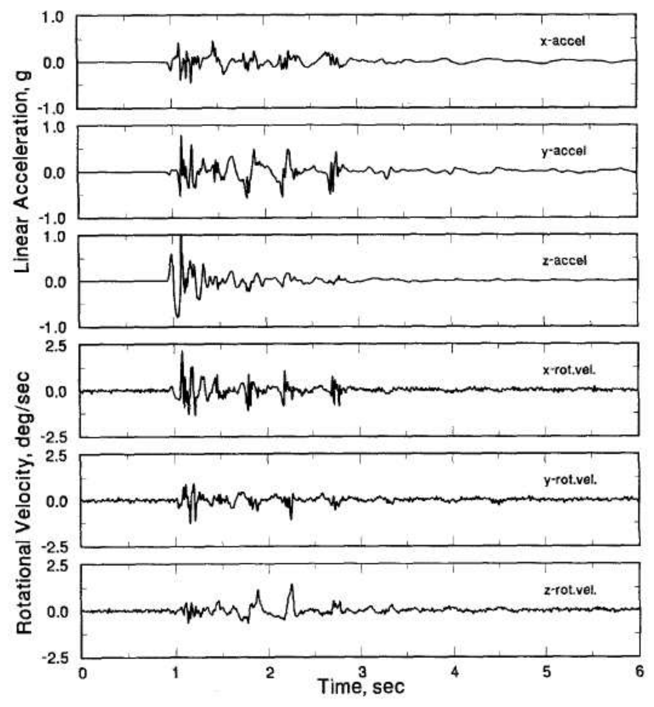
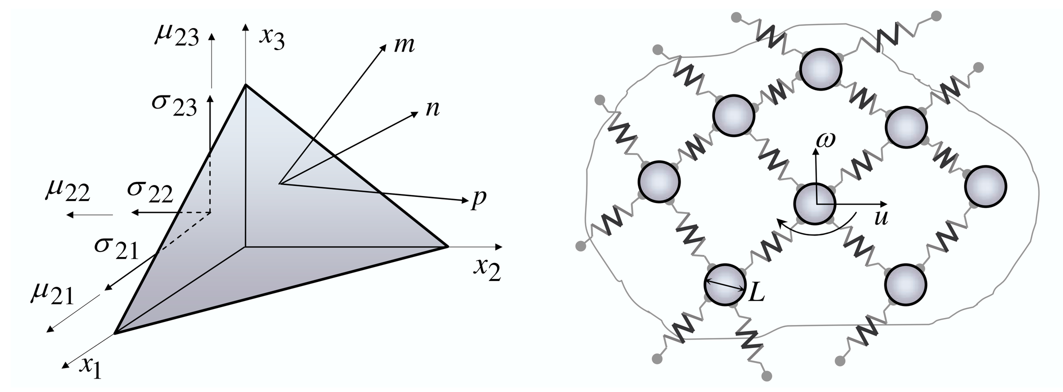


Figure 1: Measured 6DOF ground-motion data for 22 September 1993 NPE event *(Nigbor R.L. (1994) Six-degree-of-freedom ground-motion measurement, Bull. Seis. Soc. Am, 84(5), 1665-1669)*

HOW CAN BE INTRODUCED THE ROTATION VECTOR?

1. Restricted rotations: $\boldsymbol{\theta} = 1/2\,\mathrm{rot}\,\boldsymbol{u}$
2. Independent rotations: $\boldsymbol{\theta}$ and $\boldsymbol{u}$ are independent

	Equation of motion	Stress tensor	Couple-stress tensor	Geometrical equations
Classical elasticity theory	$(2\mu+\lambda)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - \mu\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{u} + \boldsymbol{X} = \rho\ddot{\boldsymbol{u}}$	$\tilde\sigma = 2\mu\tilde\varepsilon + \lambda I_1(\tilde\varepsilon)\tilde e$		$\tilde\varepsilon = (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)/2$
Cosserat pseudo-continuum	$\mu\nabla^2\boldsymbol{u} + (\mu+\lambda)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + \frac{1}{4}(\gamma+\varepsilon)\mathrm{rot}\,\mathrm{rot}\,\nabla^2\boldsymbol{u} + \boldsymbol{X} = \rho\ddot{\boldsymbol{u}}$	$\tilde\sigma = 2\mu\tilde\gamma^{(A)} + \lambda I_1(\tilde\gamma)\tilde e - \frac{1}{2}\nabla\cdot\tilde\mu\cdot\tilde{\boldsymbol{E}}$	$\tilde\mu = 2\gamma\tilde\chi^{(S)} + 2\varepsilon\tilde\chi^{(A)}$	$\tilde\gamma = \nabla\boldsymbol{u} - \tilde{\boldsymbol{E}}\cdot\boldsymbol{\theta}$, $\tilde\chi = \nabla\boldsymbol{\theta}$
Reduced Cosserat continuum	$(\lambda+2\mu)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - (\mu+\alpha)\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{u} + 2\alpha\,\mathrm{rot}\,\boldsymbol{\theta} + \boldsymbol{X} = \rho\ddot{\boldsymbol{u}}$, $2\alpha\,\mathrm{rot}\,\boldsymbol{u} - 4\alpha\boldsymbol{\theta} + \boldsymbol{Y} = j\ddot{\boldsymbol{\theta}}$	$\tilde\sigma = 2\mu\tilde\gamma^{(S)} + 2\alpha\tilde\gamma^{(A)} + \lambda I_1(\tilde\gamma)\tilde e$		$\tilde\gamma = \nabla\boldsymbol{u} - \tilde{\boldsymbol{E}}\cdot\boldsymbol{\theta}$
Cosserat continuum	$(\lambda+2\mu)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - (\mu+\alpha)\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{u} + 2\alpha\,\mathrm{rot}\,\boldsymbol{\theta} + \boldsymbol{X} = \rho\ddot{\boldsymbol{u}}$, $(\beta+2\gamma)\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\theta} - (\gamma+\varepsilon)\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\theta} + 2\alpha\,\mathrm{rot}\,\boldsymbol{u} - 4\alpha\boldsymbol{\theta} + \boldsymbol{Y} = j\ddot{\boldsymbol{\theta}}$	$\tilde\sigma = 2\mu\tilde\gamma^{(S)} + 2\alpha\tilde\gamma^{(A)} + \lambda I_1(\tilde\gamma)\tilde e$	$\tilde\mu = 2\gamma\tilde\chi^{(S)} + 2\varepsilon\tilde\chi^{(A)} + \beta I_1(\tilde\chi)\tilde e$	$\tilde\gamma = \nabla\boldsymbol{u} - \tilde{\boldsymbol{E}}\cdot\boldsymbol{\theta}$, $\tilde\chi = \nabla\boldsymbol{\theta}$



- $\boldsymbol{p} = \boldsymbol{n}\cdot\tilde\sigma$ and $\boldsymbol{m} = \boldsymbol{n}\cdot\tilde\mu$ are force and moment of force
- $\boldsymbol{u} = \{u_x, u_y, u_z\}$ is displacement vector and $\boldsymbol{\theta} = \{\theta_x, \theta_y, \theta_z\}$ is rotation vector
- μ, λ are the Lame constants,
- $\alpha, \beta, \gamma, \varepsilon$ are the physical constants of a material in the framework of the Cosserat medium,
- ρ is the density, j is the moment of inertia density,
- $(.)^{(S)}$ and $(.)^{(A)}$ denote the symmetric and antisymmetric parts of tensor, $\tilde{\boldsymbol{E}}$ is the Levi-Civita tensor of the third rank and $\tilde e$ is the identity tensor

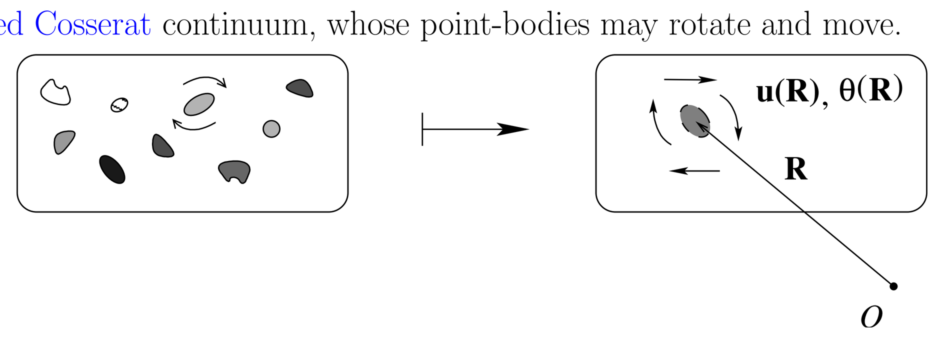
1 MOTIVATION

"The model of the classical theory of elasticity agrees well with experiments conducted on construction materials at stresses within the limit of elasticity. Appreciable differences between the theory and experiment occur in the cases where stress gradients are essential ..., in vibration problems of wave propagation and forced high-frequency vibrations ... and in granular materials"

Witold Nowacki

2 ROCKS AND COMPRESSED SOILS AS REDUCED COSSERAT CONTINUUM

We consider a heterogeneous elastic medium with inclusions as a homogeneous reduced Cosserat continuum, whose point-bodies may rotate and move.



Rotations and translations are independent. The medium reacts on the rotation of a point-body relatively to the background continuum, but there is no "rotational spring" trying to reduce the relative turn of point-bodies
$\Longrightarrow$ stress tensor is asymmetric, but couple-stress tensor is zero.

In the case $\boldsymbol{X} = 0$ and $\boldsymbol{Y} = 0$ (no external loads) we have the equations of motion as follows:

$$(\lambda+2\mu)\nabla\nabla\cdot\boldsymbol{u} - (\mu+\alpha)\nabla\times(\nabla\times\boldsymbol{u}) + 2\alpha\nabla\times\boldsymbol{\theta} = \rho\ddot{\boldsymbol{u}},$$
$$2\alpha\nabla\times\boldsymbol{u} - 4\alpha\boldsymbol{\theta} = j\ddot{\boldsymbol{\theta}}.$$

3 DYNAMIC PROBLEM OF PLANE WAVE PROPAGATION

- Representation of the solution for displacement and rotation vectors components

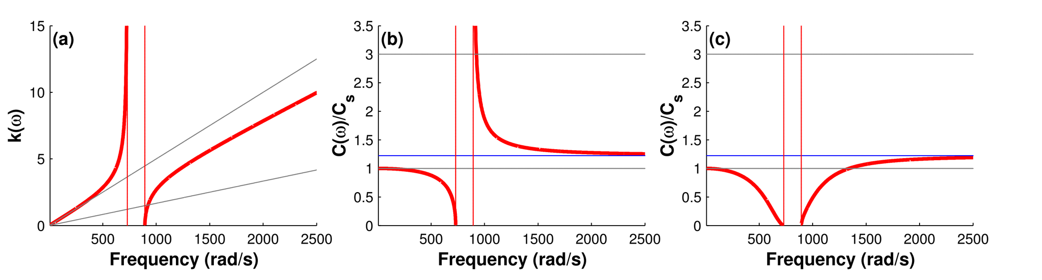
$$u_n(x,z,t) = \int_{-\infty}^{\infty} U_n(z)e^{i(kx+\omega t)}\hat u_0(\omega)\,d\omega$$
$$\theta_n(x,z,t) = \int_{-\infty}^{\infty} \theta_n(z)e^{i(kx+\omega t)}\hat u_0(\omega)\,d\omega$$

where k is the wavenumber, ω is the circular frequency, t is the time, $U_n(z)$ and $\theta_n(z)$ are amplitude functions depending on depth and frequency. $\hat u_0(\omega)$ is the complex spectral function corresponding to the Fourier spectrum of a source signal and determines the wavepacket form.

- Plane P-wave: $U_x(z) = U_x \Longrightarrow$
Dispersion relation is $k(\omega) = \omega/C_p$, where $C_p = \sqrt{\frac{\lambda+2\mu}{\rho}}$.
They are not any differencies between classical end reduced Cosserat theories.
- Plane S-wave: $U_y(z) = U_y$, $U_z(z) = U_z$, $\theta_y(z) = \theta_y$, $\theta_z(z) = \theta_z \Longrightarrow$
Dispersion relation is

$$\omega^4 - \omega^2(\omega_0^2 + C_{s\alpha}^2k^2) + \omega_0^2C_s^2k^2 = 0,$$
$$k(\omega) = \frac{\omega}{C_s}\sqrt{\frac{1-\omega^2/\omega_0^2}{1-\omega^2/\omega_1^2}},$$
$$\theta_y = \frac{ik\omega_0^2}{2(\omega^2-\omega_0^2)}U_z, \quad \theta_z = \frac{ik\omega_0^2}{2(\omega^2-\omega_0^2)}U_y, \quad \boldsymbol{\theta} \neq 1/2\,\mathrm{rot}\,\boldsymbol{u}$$

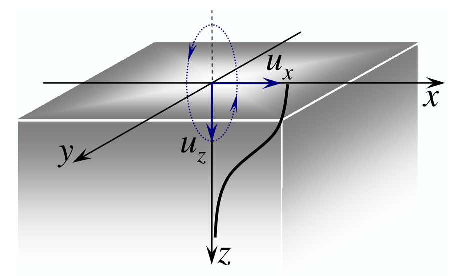
Figure 2: Example of dispersion curves of S-wave. $C_s = 200$, $C_{s\alpha} = 244.9$, $C_p = 600$, $\omega_0 = 894.4$, $\omega_1 = 730.2$. Red bold lines describe the solution for reduced Cosserat medium, red vertical lines correspond to frequencies ω_0 and ω_1, grey horizontal lines correspond to velocities C_s and C_p and blue horizontal line shows the $C_{s\alpha}$ velocity.



Here $C_s = \sqrt{\frac{\mu}{\rho}}$, $\quad C_{s\alpha} = \sqrt{\frac{\alpha+\mu}{\rho}}$, $\quad \omega_0 = 2\sqrt{\frac{\alpha}{j}}$, $\quad \omega_1 = \frac{C_s}{C_{s\alpha}}\omega_0 = \sqrt{\frac{\alpha\mu}{j(\alpha+\mu)}}$.

4 SURFACE RAYLEIGH WAVE

- Plane surface wave in the free elastic half-space is considered.
z (axis $\boldsymbol{i}_3$) — depth, x, y (axes $\boldsymbol{i}_1, \boldsymbol{i}_2$) — plane co-ordinates.



- Rayleigh-type solution decreases with depth z:

$$U_x(z) = D_1ike^{-\nu_1z} + D_2\nu_2e^{-\nu_2z}, \quad U_z(z) = -D_1\nu_1e^{-\nu_1z} + D_2ike^{-\nu_2z},$$
$$\theta_y(z) = D_2\frac{\omega^2}{2C_s^2(1-\omega^2/\omega_1^2)}e^{-\nu_2z}, \quad \boldsymbol{\theta} \neq 1/2\,\mathrm{rot}\,\boldsymbol{u}$$

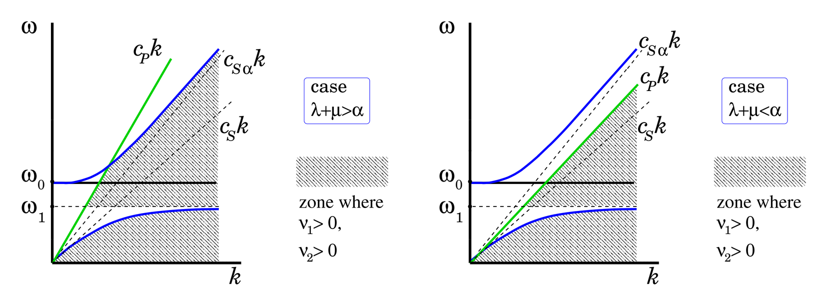
where

$$\nu_1 = \sqrt{k^2 - \frac{\omega^2}{C_p^2}} \quad \text{and} \quad \nu_2 = \sqrt{k^2 - \frac{\omega^2}{C_s^2}\left(\frac{1-\omega^2/\omega_0^2}{1-\omega^2/\omega_1^2}\right)}.$$

- Boundary conditions on the free surface is $\boldsymbol{i}_3\cdot\tilde\sigma = \boldsymbol{0} \Longrightarrow$
we obtain the dispersion relation for the Rayleigh wave:

$$4k^2\nu_1\nu_2 = (2k^2 - \omega^2/C_s^2)^2.$$

- Zones of existence for Rayleigh waves: $\nu_1, \nu_2 \in \Re$. In the classical case the Rayleigh wave exists for all ω.



- Solution of the Rayleigh equation:

$$k(\omega) = \frac{\omega}{C_s}\sqrt{\frac{\sqrt[3]{4}c(\omega)}{12a(\omega)} - \frac{b(\omega)}{6a(\omega)} + \frac{\sqrt[3]{2}(b(\omega)^2 - 12a(\omega))}{6a(\omega)c(\omega)}}.$$

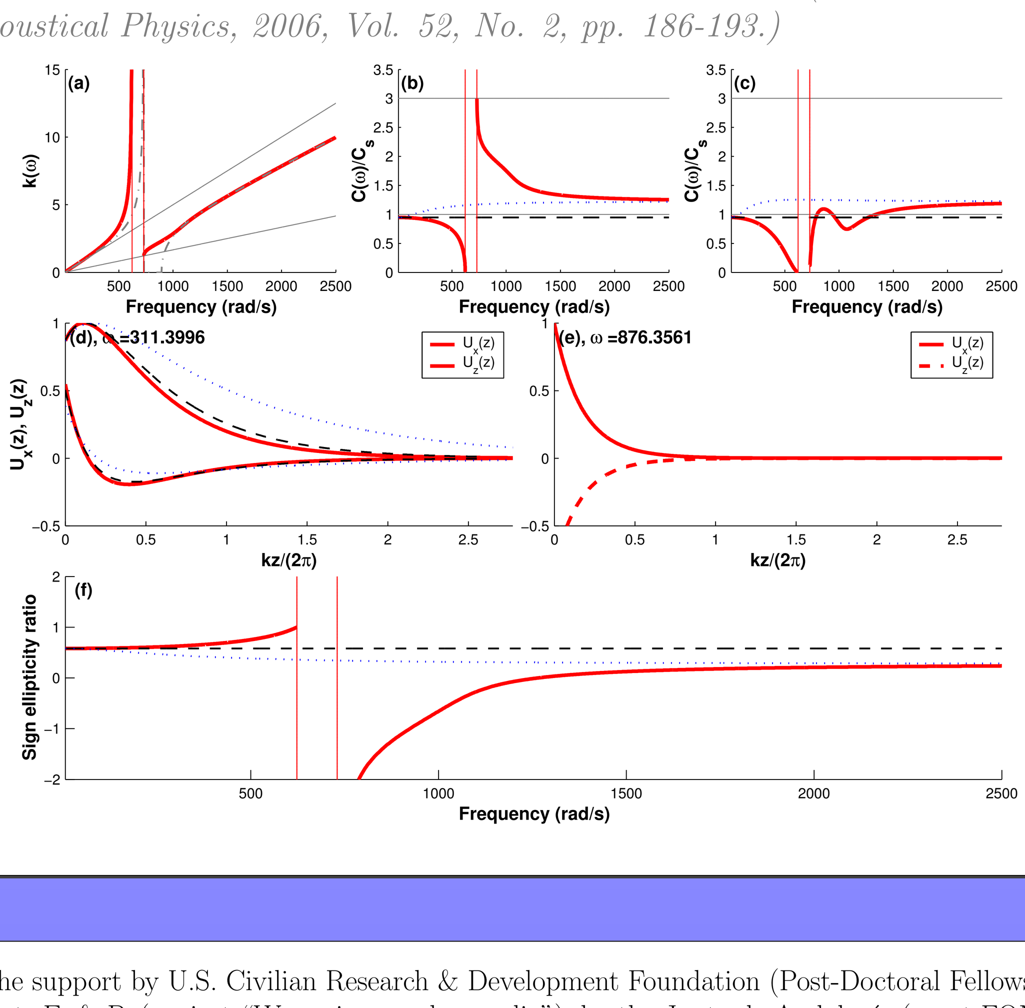
where

$$a(\omega) = 2\left(1 - \frac{C_s^2}{C_p^2} + \frac{\omega^2}{(1+\mu/\alpha)(\omega^2-\omega_1^2)}\right), \quad b(\omega) = 2\left(2\frac{C_s^2}{C_p^2} - 3 - \frac{2C_s^2\omega^2}{C_p^2(1+\mu/\alpha)(\omega^2-\omega_1^2)}\right),$$

$$c(\omega) = \sqrt[3]{27a(\omega)^2 - 2b(\omega)^3 + 36a(\omega)b(\omega) + d(\omega)},$$

$$d(\omega) = 3\sqrt{3}a(\omega)\sqrt{27a(\omega)^2 - 4b(\omega)^3 - 16b(\omega)^2 + 72a(\omega)b(\omega) + 256a(\omega)}.$$

Figure 3: Example of dispersion curves of Rayleigh waves. $C_s = 200$, $C_{s\alpha} = 244.9$, $C_p = 600$, $\omega_1 = 730.2$, $\omega_2 = 622.7$. Red bold lines describe the solution for reduced Cosserat medium, red vertical lines correspond to frequencies ω_1 and ω_2, grey horizontal lines correspond to velocities C_s and C_p, hatch black lines depict the solution for Rayleigh wave in the classical case, dotted blue lines show the solution within the wramework of Cosserat continumm *(M. Kulesh et al. Acoustical Physics, 2006, Vol. 52, No. 2, pp. 186-193.)*



5 PLANE WAVES FROM DYNAMIC POINT SOURCE

- External loads at $\boldsymbol{r} = \boldsymbol{0}$: force $\boldsymbol{X}(\boldsymbol{r},t) = \boldsymbol{X}_0\delta(\boldsymbol{r})e^{i\omega t}$, torque $\boldsymbol{Y}(\boldsymbol{r},t) = \boldsymbol{Y}_0\delta(\boldsymbol{r})e^{i\omega t}$.
- General solution:

$$\boldsymbol{u} = \frac{\boldsymbol{Y}_0e^{i\omega t}\tilde e\times\hat{\boldsymbol{r}}}{8\pi\alpha\mu r^2}\left(1-\frac{\omega^2}{\omega_1^2}\right)^{-1}\left(1+i\frac{\omega f(\omega)}{C_s}r\right)e^{-i\frac{\omega f(\omega)}{C_s}r} - \frac{\boldsymbol{X}_0e^{i\omega t}}{4\pi\mu(\lambda+2\mu)}\left(\frac{\omega^2}{C_p^2} - \frac{\omega^2f^2(\omega)}{C_s^2}\right)^{-1}\left(1-\frac{\omega^2}{\omega_1^2}\right)^{-1}\cdot$$
$$\Bigg\{-e^{-i\frac{\omega}{C_p}r}\left((\lambda+\mu-\alpha)\left(1-\frac{\omega^2}{\omega_0^2}\right)+\alpha\right)\left(\frac{1}{r}\frac{\omega^2}{C_p^2}\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + \left(1+i\frac{\omega}{C_p}r\right)\frac{\tilde e - 3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{r^3}\right)+$$
$$e^{-i\frac{\omega f(\omega)}{C_s}r}\left(\frac{1}{r}\tilde e(\lambda+2\mu)\left(1-\frac{\omega^2}{\omega_0^2}\right)\left(\frac{\omega^2}{C_p^2} - \frac{\omega^2f^2(\omega)}{C_s^2}\right)+\right.$$
$$\left.\left((\lambda+\mu-\alpha)\left(1-\frac{\omega^2}{\omega_0^2}\right)+\alpha\right)\left(\frac{1}{r}\frac{\omega^2f^2(\omega)}{C_s^2}\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + \left(1+i\frac{\omega f(\omega)}{C_s}r\right)\frac{\tilde e - 3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{r^3}\right)\right)\Bigg\},$$
$$\boldsymbol{\theta} = e^{i\left(\omega t - \frac{\omega f(\omega)}{C_s}r\right)}\left(1-\frac{\omega^2}{\omega_1^2}\right)^{-1}\Bigg\{-\frac{\boldsymbol{X}_0}{8\pi\mu}\frac{\tilde e\times\hat{\boldsymbol{r}}}{r^2}\left(1+i\frac{\omega f(\omega)}{C_s}r\right) - \frac{\boldsymbol{Y}_0}{16\pi\alpha^2\mu}\left(1-\frac{\omega^2}{\omega_0^2}\right)^{-1}\cdot$$
$$\left(\frac{1}{r}\tilde e\left(1-\frac{\omega^2}{\omega_0^2}\right)\left(\frac{\omega^2}{C_{s\alpha}^2} - \frac{\omega^2f^2(\omega)}{C_s^2}\right)(\mu+\alpha) + \frac{1}{r}\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}\frac{\omega^2f^2(\omega)}{C_s^2}\alpha + \frac{(\tilde e - 3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}})\alpha}{r^3}\left(1+i\frac{\omega f(\omega)}{C_s}r\right)\right)\Bigg\},$$

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$, $r = |\boldsymbol{r}|$, $f(\omega) = \sqrt{(1-\omega^2/\omega_0^2)/(1-\omega^2/\omega_1^2)}$, if $\omega > \omega_0$ or $\omega < \omega_1$ and $f(\omega) = -i\sqrt{(1-\omega^2/\omega_0^2)/|1-\omega^2/\omega_1^2|}$, if $\omega_1 < \omega < \omega_0$.

- Zone $\omega_1 < \omega < \omega_0$: a part of the wave does not propagate (exponential decaying in space), its energy is stored near the source. This indicates the possibility of localisation phenomena.
- First critical frequency $\omega = \omega_0$: Resonant solution for $\boldsymbol{\theta}$ if $\boldsymbol{Y}_0 \neq 0$, part of the translation wave does not propagate:

$$\boldsymbol{u} = -\frac{\boldsymbol{Y}_0e^{i\omega_0t}}{8\pi\alpha^2}\cdot\frac{\tilde e\times\hat{\boldsymbol{r}}}{r^2} - \frac{\boldsymbol{X}_0e^{i\omega_0t}}{4\pi\rho\omega_0^2}\left\{\frac{\tilde e - 3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{r^3} - e^{-i\frac{\omega_0}{C_p}r}\left(\frac{1}{r}\frac{\omega_0^2}{C_p^2}\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + \left(1+i\frac{\omega_0}{C_p}r\right)\frac{\tilde e - \hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{r^3}\right)\right\}.$$

- Second critical frequency $\omega = \omega_1$:

$$\boldsymbol{u} = -\frac{1}{4\pi}\frac{\boldsymbol{X}_0C_s^2}{\mu\omega_1^2}e^{i\omega_1(t-r/C_p)}\left(\frac{1}{r}\frac{\omega_1^2}{C_p^2}\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + (1+i\frac{\omega_1}{C_p})\frac{\tilde e - 3\hat{\boldsymbol{r}}\hat{\boldsymbol{r}}}{r^3}\right) - \frac{C_{S\alpha}^2}{2\alpha\omega_1^2}\nabla\times\boldsymbol{Y}_0\,\delta(\boldsymbol{r})e^{i\omega_1t},$$
$$\boldsymbol{\theta} = e^{i\omega_1t}\frac{\mu+\alpha}{4\alpha^2}\left(\boldsymbol{Y}_0\delta(\boldsymbol{r}) + \frac{C_{S\alpha}^2}{\omega_1^2}\boldsymbol{Y}_0\triangle\delta(\boldsymbol{r}) - \frac{C_{S\alpha}^2}{\omega_1^2}\nabla\nabla\cdot\boldsymbol{Y}_0\delta(\boldsymbol{r})\right)$$

In case if $\boldsymbol{Y}_0 = \boldsymbol{0}$, we have $\boldsymbol{\theta} = \boldsymbol{0}$, and in the translational displacement waves we see the only non-zero part propagating with the phase velocity C_p.

- Third critical frequency $\omega^2f^2(\omega) = C_s^2/C_p^2$: intersection of S- and P- dispersion graphs for 3D plane waves, resonant solution for $\boldsymbol{u}$.
- Particular case $\boldsymbol{Y}_0 = \boldsymbol{0}$. At this limit the medium behaves as a classical elastic medium with corresponding constants: for $\omega << \omega_1$ with phase velocities C_p and C_s and for $\omega >> \omega_0$ with phase velocities C_p and $C_{s\alpha}$.

6 CONCLUSIONS

- We have considered the reduced Cosserat medium where the couple stress is zero, while the rotation vector is independent of the translational displacement: $\boldsymbol{\theta} \neq 1/2\,\mathrm{rot}\,\boldsymbol{u}$.
- In this model, the stress depends on the rotation of a particle relatively to the background continuum of mass centers, but it does not depend on the relative rotation of two neighboring particles.
- We have obtained and analyzed theoretical solutions for this model which describe the propagation of plane P-waves, S-waves, surface Rayleigh waves and reaction of the medium to the dynamic point source.
- We have shown both the dispersive character of these waves in elastic space and half space, and the existence of forbidden frequency zones.
- There is a zone $(\omega_1; \omega_0)$ where the S-wave does not propagate (similar to the 3D case; unlike the classical and full Cosserat medium cases).
- Waves produced by a dynamic point source with the frequency in the zone $(\omega_1; \omega_0)$ partially do not propagate (indicates possibility of localisation).
- There is a forbidden zone $(\omega_2; \omega_1)$ where the Rayleigh wave does not propagate (similar to the 3D case; unlike the classical and full Cosserat medium cases).
- The energy of wave in forbidden frequency zone is caught by rotation and localised.
- There is not surface transversal wave decreasing with depth (similar to the classical case, unlike the full Cosserat case).
- These results can be used for the preparation, execution, and interpretation of seismic experiments, which would allow one to determine whether (and in which situations) polar theories are important in rock mechanics, and to help with the identification of material parameters of the reduced Cosserat continuum.

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