# Nonlinear isotropic elastic reduced Cosserat continuum as a possible model for geomedium. Spherical presstressed state

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# Plan of the talk

- Motivation
- Nonlinear elastic reduced Cosserat continuum: equations
- Small perturbations near a nonlinear prestressed state for a material with a strain energy of general kind (smooth enough)
- Small perturbations near a spherical prestressed state:
  - for an isotropic material
  - for an isotropic physically linear material: waves, localisation, and instabilities
- Axially symmetric prestressed state: anisotropic effects

## Motivation

There are several scales on which we can consider ground motions. **Rotations** may be important on many scales.

- wave propagation in earthquakes: heterogeneities, solid blocks in soil and rock may have their proper rotational dynamics due to wave scattering
- motion of large blocks in geomedium: slow rotation, vortex-like structures, loss of stability in shear-rotational motion

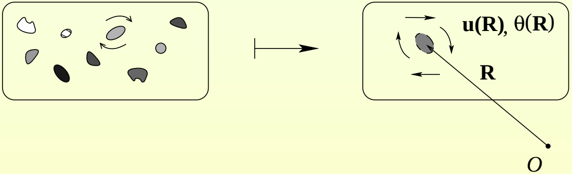
**Nonlinearity** is always present in nature for large motions. Well-known experimental evidences:

- nonlinearity of soils
- "induced", or "local" anisotropy of soils: anisotropy or prestressed state?

I suggest a rigorous model. You have to decide if it is useful.

# Nonlinear elastic reduced Cosserat continuum

Modelling of a rock:
We consider a heterogeneous elastic medium with inclusions as a homogeneous reduced Cosserat continuum, whose point-bodies may rotate and move.




Modelling of a compressed soil: An "averaged" particle may rotate and move, the surrounding medium resists to its rotation and motion.

**In both cases there is no ordered structure of rotations**

## References

**Eringen, Kafadar**: nonlinear full elastic Cosserat continuum
**Zhilin**: the method to obtain its constitutive equations via the law of balance of energy
Many works on Cosserat continua: Green, Naghdi, Rivlin, Erbay, Suhubi, Nowacki, Palmov, Aero,...
Rotation in granular media (modelling as full Cosserat continuum): Vardoulakis, Besdo, Metrikine, Askes, Suiker, Rene de Borst
Granular media as linear reduced isotropic Cosserat continuum: **Schwartz, Johnson, Feng**
Waves in linear elastic reduced Cosserat continuum: Grekova, Herman, Kulesh

## Cosserat continuum. Stress and couple stress tensors

If $\mathbf{t}_{(\mathbf{n})}$ is a force acting upon an infinitesimal surface with normal $\mathbf{n}$, there exist a tensor $\boldsymbol{\tau}$ such that

$$\mathbf{t}_{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\tau} \tag{1}$$

Then $\boldsymbol{\tau}$ is a Cauchy stress tensor. In classical medium, where particles have no rotational inertia, $\boldsymbol{\tau}$ is symmetric. It follows from the balance of moment. In Cosserat media balance of moment has other terms and $\boldsymbol{\tau} \neq \boldsymbol{\tau}^{\top}$.

If $\mathbf{m}_{(\mathbf{n})}$ is a moment acting upon an infinitesimal surface with normal $\mathbf{n}$, there exist a tensor $\boldsymbol{\mu}$ such that

$$\mathbf{m}_{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\mu} \tag{2}$$

Then $\boldsymbol{\mu}$ is a couple stress tensor.

$\boldsymbol{\tau}$ works on $\nabla \mathbf{v}$ (gradient of translational velocity), and $\boldsymbol{\omega}$ (angular velocity); $\boldsymbol{\mu}$ works on $\nabla \boldsymbol{\omega}$ (gradient of angular velocity).

# Nonlinear elastic reduced Cosserat continuum. Balance of energy

**Reduced** Cosserat medium: **rotations and translations are kinematically independent**, as in the full Cosserat continuum. The medium reacts on the rotation of a particle relatively to the background continuum, but there is **no "rotational spring" trying to reduce the relative turn of particles**

balance of energy $\Longrightarrow$

$$\rho \dot{U} = \boldsymbol{\tau}^{\top} \cdot\cdot \nabla \mathbf{v} - \boldsymbol{\tau}_{\times} \cdot \boldsymbol{\omega} + \boldsymbol{\mu}^{\top} \cdot\cdot \nabla \boldsymbol{\omega}$$

the Cauchy stress tensor $\boldsymbol{\tau}$ is asymmetric, but
the couple stress $\boldsymbol{\mu}$ is zero
$\rho$ is the mass density, $U$ is the strain energy, $\mathbf{v}$ is the translational velocity, $\boldsymbol{\omega}$ is the rotational velocity, $\boldsymbol{\tau}_{\times} = \tau_{mn} \mathbf{i}_m \times \mathbf{i}_n$.

# General ideas

We will obtain equations for the nonlinear elastic reduced Cosserat continuum basing upon

- balance of force
- balance of torque (moment)
- balance of energy
- principle of material objectivity
- 2nd law of thermodynamics — satisfied automatically for elastic materials
- symmetry considerations

## Nonlinear elastic reduced Cosserat continuum. Kinematic characteristics

In a nonlinear medium we have consider a reference configuration (e.g. natural or initial state) and the actual one (current state). Each point of a Cosserat is enriched with a triad $\mathbf{D}_k$ (actual configuration), $\mathbf{d}_k$ in the reference one. Let be $\overset{\circ}{\nabla}$ the nabla operator in the reference configuration, $\mathbf{R}$ and $\mathbf{P}$ the radius vector and the rotation tensor in the actual configuration, respectively, $\mathbf{A} = \overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{P}$ (an objective strain tensor).
Rotation tensor turns the triad (by definition): $\mathbf{P} \cdot \mathbf{d}_k = \mathbf{D}_k$ . This gives $\mathbf{P} = \mathbf{D}_m \otimes \mathbf{d}^m$.

# Nonlinear elastic reduced Cosserat continuum. Constitutive equations

Balance of energy:

$$\rho \dot{U} = \boldsymbol{\tau}^{\top} \cdot\cdot \nabla \mathbf{v} - \boldsymbol{\tau}_{\times} \cdot \boldsymbol{\omega} = (\overset{\circ}{\nabla} \mathbf{R}^{-\top} \cdot \boldsymbol{\tau} \cdot \mathbf{P})^{\top} \cdot\cdot \dot{\mathbf{A}} \qquad \overset{\text{if } U = U(\mathbf{A})}{\Longrightarrow}$$

$$\boldsymbol{\tau} = \rho \, \overset{\circ}{\nabla} \mathbf{R}^{\top} \cdot \frac{\partial U}{\partial \mathbf{A}} \cdot \mathbf{P}^{\top}$$ — constitutive equation

This we rewrite for convenience as

$$\mathbf{T} = \rho_0 \frac{\partial U}{\partial \mathbf{A}} \cdot \mathbf{P}^{\top},$$

where $\mathbf{T} \overset{\text{def}}{=} \overset{\circ}{\nabla} \mathbf{R}^{-\top} \cdot \boldsymbol{\tau} \det \overset{\circ}{\nabla} \mathbf{R}$ — first Piola-Kirchoff stress tensor.

# Nonlinear elastic reduced Cosserat continuum

Laws of dynamics (balances of linear and angular momenta)

$$\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{K} = \rho \dot{\mathbf{v}} \qquad \overset{\circ}{\nabla} \cdot \mathbf{T} + \rho_0 \mathbf{K} = \rho_0 \dot{\mathbf{v}}, \tag{3}$$

$$\boldsymbol{\tau}_\times + \rho \mathbf{L} = \rho \mathbf{I} \cdot \dot{\boldsymbol{\omega}} \qquad [\overset{\circ}{\nabla} \mathbf{R}^\top \cdot \mathbf{T}]_\times + \rho_0 \mathbf{L} = \rho_0 \mathbf{I} \cdot \dot{\boldsymbol{\omega}} \tag{4}$$

where $\mathbf{K}, \mathbf{L}$ are mass densities of force and torque, $\mathbf{v}$ and $\boldsymbol{\omega}$ are translational and angular velocities, respectively, $\rho$ is the mass density, $\mathbf{I}$ is the density of the tensor of inertia, $\boldsymbol{\tau}_\times \overset{\text{def}}{=} \tau_{mn} \mathbf{i}^m \times \mathbf{i}^n$. Then in absence of volume torques in (quasi)statics the stress tensor is symmetric.

## Nonlinear elastic reduced Cosserat continuum: why?

In (quasi)statics, and in absence of volume torques, the stress tensor is symmetric as in the classical medium. However, there are experiments (discussion with S. Slotterback at ESMC 2009) that show that in soils, even under quasistatic loading, dynamic processes occur when shear takes place, and large rotations are present in the shear band.

Our interpretation: The medium reaches quasistatically a certain nonlinear state, where the instability with respect to shear-rotation occurs. This rotational-shear unstable motion yields in rearrangement of system of contacts in the medium, and may cause dilatation or compression due to repacking.

Shear-compression coupling in stable situation can be also provided by anisotropy induced by nonlinear stress state, or by natural infinitesimal anisotropy of the medium.

# Small deviations from the nonlinear equilibrium

Consider $U(\mathbf{A})$ twice differentiable.

Nonlinear equilibrium: $\boldsymbol{\tau} = \boldsymbol{\tau}_s, \quad \mathbf{T} = \mathbf{T}_s, \quad \mathbf{A} = \mathbf{A}_s, \quad \mathbf{R} = \mathbf{R}_s.$

Small deviations from this state:

$\mathbf{R} = \mathbf{R}_s + \mathbf{u}, \quad \mathbf{P} = (\mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}) \cdot \mathbf{P}_s.$

We calculate strain tensor in the perturbed state

$$\mathbf{A} = \mathbf{A}_s + (\nabla \mathbf{u} + \overset{\circ}{\nabla} \mathbf{R}_s \times \boldsymbol{\theta}) \cdot \mathbf{P}_s + o^2(1)$$

and 1st Piola-Kirchoff stress tensor in the perturbed state (using the constitutive equation):

$$\mathbf{T} = \mathbf{T}_s - \mathbf{T}_s \times \boldsymbol{\theta} + \rho_0 \left( \left( \left[ \frac{\partial^2 U}{\partial \mathbf{A}^2} \right]_s \cdot {\mathbf{P}_s}^\top \right) \cdot\cdot (\nabla \mathbf{u} - \boldsymbol{\theta} \times \mathbf{F}_s) \right) \cdot {\mathbf{P}_s}^\top + o^2(1),$$

where

$$\mathbf{F}_s = [\overset{\circ}{\nabla} \mathbf{R}]_s{}^\top, \quad \mathbf{T}_s = \rho_0 \left[ \frac{\partial U}{\partial \mathbf{A}} \right]_s \cdot {\mathbf{P}_s}^\top.$$

# Small deviations from the nonlinear equilibrium. Dynamics

First law of dynamics of Euler (balance of force):

$$\overset{\circ}{\nabla} \cdot (\rho_0 (( \left[ \frac{\partial^2 U}{\partial \mathbf{A}^2} \right]_s \cdot \mathbf{P}_s{}^\top) \cdot\cdot (\mathbf{u} \overset{\circ}{\nabla} - \boldsymbol{\theta} \times \mathbf{F}_s)) \cdot \mathbf{P}_s{}^\top) - \overset{\circ}{\nabla} \cdot \mathbf{T}_s \times \boldsymbol{\theta} = \rho_0 \ddot{\mathbf{u}}$$

Second law of dynamics of Euler (balance of torque):

$$\mathbf{u} \times (\overset{\circ}{\nabla} \cdot \mathbf{T}_s) + \boldsymbol{\theta}\, \mathbf{F}_s \cdot\cdot \mathbf{T}_s - \boldsymbol{\theta} \cdot \mathbf{F}_s \cdot \mathbf{T}_s + [\rho_0 ((\mathbf{F}_s \cdot \left[ \frac{\partial^2 U}{\partial \mathbf{A}^2} \right]_s \cdot \mathbf{P}_s{}^\top) \cdot\cdot (\mathbf{u} \overset{\circ}{\nabla} - \boldsymbol{\theta} \times \mathbf{F}_s)) \cdot \mathbf{P}_s{}^\top]_\times = \rho_0 \mathbf{I} \cdot \ddot{\boldsymbol{\theta}}$$

## Spherical stress state in an **isotropic** material

Isotropy: spherical deformation $\mathbf{A}_s = A\mathbf{E}$, spherical stress state $\mathbf{T}_s = T\mathbf{E}$, $\boldsymbol{\tau}_s = (T/A^2)\mathbf{E}$, tensor of elastic constants

$$\rho_0 \left[ \frac{\partial^2 U}{\partial \mathbf{A}^2} \right]_s = \mathbf{X} = \lambda_s \mathbf{E}\mathbf{E} + 2\mu_s (\mathbf{i}_m \mathbf{i}_n)^S (\mathbf{i}^m \mathbf{i}^n)^S + 2\alpha_s (\mathbf{i}_m \mathbf{i}_n)^A (\mathbf{i}^n \mathbf{i}^m)^A,$$

where $\lambda_s, \mu_s, \alpha_s$ depend on the stress state. Denote $\boldsymbol{\varphi} = A\boldsymbol{\theta}$. Then equations of motion are similar to those of the reduced isotropic linear Cosserat continuum:

$$(\lambda' + 2\mu') \mathring{\nabla} \mathring{\nabla} \cdot \mathbf{u} - (\mu' + \alpha') \mathring{\nabla} \times (\mathring{\nabla} \times \mathbf{u}) + 2\alpha' \mathring{\nabla} \times \boldsymbol{\varphi} = \rho_0 \ddot{\mathbf{u}}$$

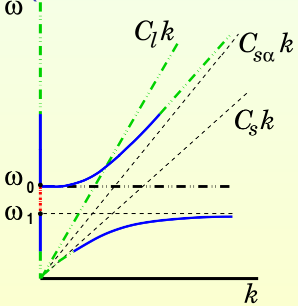
$$2\alpha' \mathring{\nabla} \times \mathbf{u} - 4\alpha' \boldsymbol{\varphi} = (\rho_0/A)\mathbf{I} \cdot \ddot{\boldsymbol{\varphi}}$$

$$\text{(for spherical tensor of inertia)} = (\rho_0 I/A) \ddot{\boldsymbol{\varphi}}$$

$$\alpha' = \alpha_s - T/(2A), \quad \mu' = \mu_s + T/(2A), \quad \lambda' + 2\mu' = \lambda_s + 2\mu_s.$$

What about the signs of the coefficients?

# Isotropic linear reduced Cosserat continuum: dispersion relation (Grekova, Herman)




$$\omega_0^2 = 4\alpha/(\rho_0 I),$$
$$\omega_1^2 = \omega_0^2/(1+\alpha/\mu),$$
$$C_s^2 = \mu/\rho,$$
$$C_{s\alpha}^2 = (\mu+\alpha)/\rho,$$
$$C_l^2 = (\lambda+2\mu)/\rho.$$

Shear dispersion relation:

$$k_S^2 = \frac{\omega^2}{C_S^2} \cdot \frac{1-\omega^2/\omega_0^2}{1-\omega^2/\omega_1^2} = \frac{\omega^2}{C_S^2} f^2(\omega).$$

# Physically linear, geometrically nonlinear Cosserat medium

$\lambda_s = \lambda, \mu_s = \mu, \alpha_s = \alpha$ do not depend on the stress state.

$$\rho_0 U = \frac{1}{2}(\mathbf{A} - \mathbf{E}) \cdot\cdot \mathbf{X} \cdot\cdot (\mathbf{A} - \mathbf{E})^\top$$

$$T = 3(A-1)(\lambda + 2\mu), \quad \alpha' = \alpha - \frac{A-1}{A}(\lambda + 2\mu), \tag{5}$$

$$\mu' = \mu + \frac{A-1}{A}(\lambda + 2\mu), \quad \lambda' + 2\mu' = \lambda + 2\mu. \tag{6}$$

Strong compression: $\mu' < 0$, the shear wave becomes instable, material fails under small shear perturbation (at rather low frequencies). Strong expansion: for $\alpha < \lambda + 2\mu$ we may have $\alpha' < 0$, mechanism of failure is via rotations (at $\omega_0$).

## Stable situation: properties of Green functions for $\lambda', \mu', \alpha' > 0$

We calculated reaction of a medium on harmonic point sources: external force $\rho\mathbf{K} = \mathbf{K}_0 e^{-i\omega t}\delta(\mathbf{r})$, external torque $\rho\mathbf{L} = \mathbf{L}_0 e^{-i\omega t}\delta(\mathbf{r})$. The expressions are tedious. If $\omega \neq \omega_1, \omega \neq \omega_0$,

$$\mathbf{u} = e^{-i\omega t}(\mathbf{K}_0 \cdot (e^{i\frac{\omega}{c_P}r}\mathbf{G}_{kup}(\mathbf{r},\omega) + e^{i\frac{\omega f(\omega)}{c_S}r}\mathbf{G}_{kus}(\mathbf{r},\omega)) + \mathbf{L}_0 \cdot e^{i\frac{\omega f(\omega)}{c_S}r}\mathbf{G}_{lus}(\mathbf{r},\omega))$$

$$\varphi = e^{-i(\omega t - \frac{\omega f(\omega)}{c_S}r)}(\mathbf{K}_0 \cdot \mathbf{G}_{k\varphi s}(\mathbf{r},\omega) + \mathbf{L}_0 \cdot \mathbf{G}_{l\varphi s}(\mathbf{r},\omega)),$$

$$f^2(\omega) = (1 - \omega^2/\omega_0^2)/(1 - \omega^2/\omega_1^2)$$

- a strong frequency dependence,
- if $\mathbf{L} \neq \mathbf{0}$, resonance at $\omega = \omega_0$, strong localisation at $\omega = \omega_1$
- exponentially decaying part of wave if $\omega \in (\omega_1; \omega_0)$.

# Localisation for $\lambda', \mu', \alpha' > 0$

If we have infinitesimal heterogeneities in density and inertia moment density $\rho = \rho_0 + \varepsilon\rho_1'$, $I = I_0 + \varepsilon I_1'$, and $\omega$ is a frequency of the incident wave, a part of wave is localised near heterogeneity if $\omega \in (\omega_1; \omega_0)$.

If the inhomogeneity is localised at the origin:
$\rho_1' = \rho_1\delta(\mathbf{r}), I_1' = I_1\delta(\mathbf{r})$. Then the correction terms for $\mathbf{u}, \boldsymbol{\varphi}$ are given by the Green function (above) with $\mathbf{K}_0 = \omega^2\mathbf{U}_0\rho_1$, $\mathbf{L}_0 = \omega^2\mathbf{\Phi}_0 I_1$, where $\mathbf{U}_0, \mathbf{\Phi}_0$ are amplitudes of the incident wave.

# Rayleigh-type wave (Kulesh, Grekova)

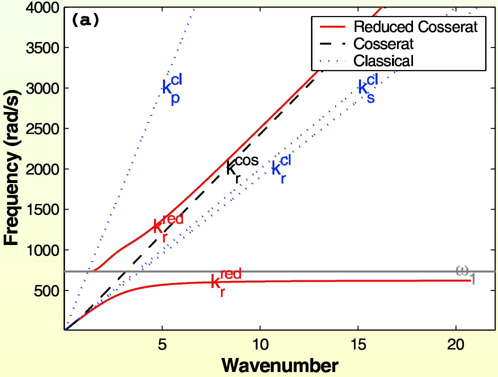



Figure: Numerical example illustrating the behavior of (a) wavenumber, (b) phase, and (c) group velocities for a Rayleigh wave in the reduced Cosserat continuum

# Rayleigh-type wave (Kulesh, Grekova)

Analytical results

1. For one dispersion branch, there exist both cut-off frequency $\omega_1$ and cut-off wave number $k_1 = \omega_1 / C_l$: this branch starts at the point $(k_1; \omega_1)$.
2. The Rayleigh wave at large frequencies can be faster than the shear wave in the unbounded 3D medium at small frequencies
3. There is a forbidden band of frequencies lying below $\omega_1$, where the Rayleigh-type wave does not exist
4. One of dispersion curves has an asymptote $\omega = \omega_2$. There is a forbidden band of frequencies lying upon $\omega_2$.

In all numerical experiments we had a forbidden zone $(\omega_2; \omega_1)$.

# Prestressed state with axial symmetry for an isotropic material

Consider a prestressed state, where $\mathbf{P} = \mathbf{E}$, $\mathbf{A} = A\mathbf{E} + A_3\mathbf{i}_3\mathbf{i}_3$, $\mathbf{T} = T\mathbf{E} + T_3\mathbf{i}_3\mathbf{i}_3$ (approximation to the triaxial test).

The balance of force and torque include anisotropic terms, but are **not** similar to the equations of anisotropic Cosserat medium:

$$(\lambda' + 2\mu')\mathring{\nabla}\mathring{\nabla} \cdot \mathbf{u} - (\mu' + \alpha')\mathring{\nabla} \times (\mathring{\nabla} \times \mathbf{u}) + 2\alpha'\mathring{\nabla} \times \boldsymbol{\varphi}$$

$$+ (\mu A_3/A)\mathbf{i}_3\mathbf{i}_3 \cdot (\mathring{\nabla} \times \boldsymbol{\varphi}) + ((\mu A_3 + T_3)/A)\mathbf{i}_3 \cdot \mathring{\nabla}\boldsymbol{\varphi} \times \mathbf{i}_3) = \rho_0\ddot{\mathbf{u}}$$

$$2\alpha'\mathring{\nabla} \times \mathbf{u} - 4\alpha'\boldsymbol{\varphi}$$

$$+ (T_3 - 2\mu A_3)\mathbf{i}_3 \cdot \mathring{\nabla}\mathbf{u}^S \times \mathbf{i}_3 + (T_3 - 2\alpha A_3)\mathbf{i}_3 \cdot \mathring{\nabla}\mathbf{u}^A \times \mathbf{i}_3$$

$$+ (T_3 + TA_3/A - A_3(4\alpha + (\mu + \alpha)A_3/A))\boldsymbol{\varphi} \cdot (\mathbf{E} - \mathbf{i}_3\mathbf{i}_3) = (\rho_0/A)\mathbf{I} \cdot \ddot{\boldsymbol{\varphi}}$$

(for spherical tensor of inertia) $= (\rho_0 I/A)\ddot{\boldsymbol{\varphi}}$

# Conclusions

- We have obtained equations for small deviations from a nonlinear equilibrium state in the elastic reduced Cosserat medium with smooth strain energy
- for a spherical prestressed state in an isotropic material these equations are similar to equations of the linear elastic reduced Cosserat continuum and inherit their properties (prohibited frequency band for the shear-rotation wave, localisation, resonant frequency, strong dispersion)
- in the physically linear material, strong compression leads to the instability via shearing, strong tension may lead to the instability via rotation
- equations for small deviations from an axially symmetric prestressed state in an isotropic material have anisotropic terms. However, they are not the equations of any linear anisotropic Cosserat continuum

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# Invitation

The Annual International conference
"Advanced Problems in Mechanics"
http://apm-conf.spb.ru
St. Petersburg, Russia
July 1–8, 2011
Traditionally many presentations on non-classical continua, wave propagations and dynamics of rigid bodies
You are welcome!